Hamiltonian cycles in $(2,3,c)$-circulant digraphs

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Abstract

Let D be the circulant digraph with n vertices and connection set $\{2,3,c\}$. (Assume D is loopless and has outdegree 3.) Work of S. C. Locke and D. Witte implies that if n is a multiple of 6, $c \in \{(n/2)+2, (n/2)+3\}$, and c is even, then D does *not* have a hamiltonian cycle. For all other cases, we construct a hamiltonian cycle in D.

1 Introduction

For $S \subset \mathbb{Z}$, the *circulant digraph* with vertex set $\mathbb{Z}_n$ and arcs from v to $v+s$ for each $v \in \mathbb{Z}_n$ and $s \in S$ is denoted $\operatorname{Circ}(n;S)$. A fundamental open problem is to determine which circulant digraphs have hamiltonian cycles. By the following elegant result, circulant digraphs of outdegree three are the smallest digraphs that need to be considered.

Theorem 1.1 (R. A. Rankin, 1948, [2, Thm. 4]) *The circulant digraph* $\operatorname{Circ}(n;a,b)$ *of outdegree 2 has a hamiltonian cycle iff there exist* $s,t \in \mathbb{Z}^+$, *such that*

- $s+t = \gcd(n, a-b)$, *and*
- $\gcd(n, sa+tb) = 1$.

S. C. Locke and D. Witte [1] found two infinite families of non-hamiltonian circulant digraphs of outdegree 3; one of the families includes the following examples.

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Theorem 1.2 (Locke-Witte, cf. [1, Thm. 1.4])

(1) $\mathrm{Circ}(6m; 2, 3, 3m+2)$ *is not hamiltonian if and only if m is even.*
(2) $\mathrm{Circ}(6m; 2, 3, 3m+3)$ *is not hamiltonian if and only if m is odd.*

In this paper, we show that the above examples are the only loopless digraphs of the form $\mathrm{Circ}(n; 2, 3, c)$ that have outdegree 3 and are not hamiltonian:

Theorem 1.3 *Assume $c \not\equiv 0, 2, 3 \pmod{n}$. The digraph $\mathrm{Circ}(n; 2, 3, c)$ is **not** hamiltonian iff all of the following hold*

(1) *n is a multiple of 6, so we may write $n = 6m$,*
(2) *either $c = 3m + 2$ or $c = 3m + 3$, and*
(3) *c is even.*

The direction ($\Leftarrow$) of Theorem 1.3 is a restatement of part of the Locke-Witte Theorem (1.2), so we need only prove the opposite direction.

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2 Preliminaries

Our goal is to establish Theorem 1.3($\Rightarrow$). We will prove the contrapositive.

Notation 2.1 *Let v_1, v_2 be vertices of $\mathrm{Circ}(6m; 2, 3, c)$ and let $s \in \{2, 3, c\}$.*

- *The arc from v_1 to $v_1 + s$ is called an s-arc.*
- *If $v_1 + s = v_2$, we use $v_1 \overset{s}{\text{---}} v_2$ to denote the s-arc from v_1 to v_2.*
- *If $v_1 + ks = v_2$ for some natural number k, we use $v_1 \overset{s}{\text{- - - -}} v_2$ to denote the path $v_1, v_1 + s, v_1 + 2s, \ldots, v_2$.*

Assumption 2.2 *Throughout the paper:*

(1) *We assume the situation of Theorem 1.3, so $n, c \in \mathbb{Z}^+$, and $c \not\equiv 0, 2, 3 \pmod{n}$.*
(2) *We may assume $c \not\equiv 1, -1 \pmod{n}$. (Otherwise, $\mathrm{Circ}(n; 2, 3, c)$ has a hamiltonian cycle consisting entirely of c-arcs.)*
(3) *Since the vertices of $\mathrm{Circ}(n; 2, 3, c)$ are elements of $\mathbb{Z}_n$, we may assume $3 < c < n$.*

(4) *We assume n is divisible by 6 and write $n = 6m$. (Otherwise,* $\mathrm{Circ}(n; 2, 3, c)$ *has either a hamiltonian cycle consisting entirely of 2-arcs or a hamiltonian cycle consisting entirely of 3-arcs.)*

Notation 2.3 *Let H be a subdigraph of* $\mathrm{Circ}(6m; 2, 3, c)$*, and let v be a vertex of H.*

(1) *We let $d^+_H(v)$ and $d^-_H(v)$ denote the number of arcs of H directed out of, and into, vertex v, respectively.*
(2) *If $d^+_H(v) = 1$, and the arc from v to $v + a$ is in H, then we say that v* travels by a in *H.*

Notation 2.4 *Let u and w be integers representing vertices of* $\mathrm{Circ}(6m; 2, 3, c)$*. If $u - 1 \leq w < u + n$, let*

$$I(u, w) = \{u, u + 1, \ldots, w\}$$

be the interval of vertices from u to w. (Note that $I(u, u) = \{u\}$ and $I(u, u - 1) = \emptyset$.)

We now treat two simple cases so that they will not need to be considered in later sections.

Lemma 2.5 *For any m,* $\mathrm{Circ}(6m; 2, 3, 6m - 2)$ *and* $\mathrm{Circ}(6m; 2, 3, 6m - 3)$ *have hamiltonian cycles.*

PROOF. The following is a hamiltonian cycle in $\mathrm{Circ}(6m; 2, 3, 6m-2)$, where we use -2 to denote the $(6m - 2)$-arc:

$$\begin{array}{ccccccccc}
0 & \xrightarrow{2} & 4 & \xrightarrow{3} & 7 & \xrightarrow{-2} & 3 & \xrightarrow{3} & 6 \\
 & \xrightarrow{2} & 10 & \xrightarrow{3} & 13 & \xrightarrow{-2} & 9 & \xrightarrow{3} & 12 \\
 & & & \vdots & & & & & \\
 & \xrightarrow{2} & 6m-8 & \xrightarrow{3} & 6m-5 & \xrightarrow{-2} & 6m-9 & \xrightarrow{3} & 6m-6 \\
 & \xrightarrow{2} & 6m-2 & \xrightarrow{3} & 1 & \xrightarrow{-2} & 6m-3 & \xrightarrow{3} & 0.
\end{array}$$

The following is a hamiltonian cycle in $\mathrm{Circ}(6m; 2, 3, 6m - 3)$, where we use -3 to denote the $(6m - 3)$-arc:

$$\begin{array}{ccccccccc}
0 & \xrightarrow{3} & 6m-6 & \xrightarrow{2} & 6m-4 & \xrightarrow{-3} & 2 & \xrightarrow{2} & 4 \\
 & \xrightarrow{3} & 6m-5 & \xrightarrow{2} & 1 & \xrightarrow{-3} & 6m-2 & \xrightarrow{2} & 0.
\end{array}$$

$\square$

3 Most cases of the proof

In this section, we prove the following two results that cover most of the cases of Theorem 1.3:

Proposition 3.1 *If* $c \leq 3m$ *and* $c \not\equiv 3 \pmod 6$*, then* $\mathrm{Circ}(6m; 2, 3, c)$ *has a hamiltonian cycle.*

Proposition 3.2 *If* $c > 3m$ *and* $c \notin \{3m+2, 3m+3\}$*, then* $\mathrm{Circ}(6m; 2, 3, c)$ *has a hamiltonian cycle.*

Notation 3.3 *For convenience, let* $c' = 6m - c$.

Note that $\mathrm{Circ}(6m; 2, 3, c) = \mathrm{Circ}(6m; 2, 3, -c')$, and thus by Assumption 2.2(2) and Lemma 2.5, we may assume $3 < c' < 6m - 3$.

Remark 3.4 *The use of* c' *is very convenient when* c *is large (so one should think of* c' *as being small — less than* $3m$*), but it can also be helpful in some other cases.*

Definition 3.5 *A subdigraph* P *of* $\mathrm{Circ}(6m; 2, 3, -c')$ *is a* pseudopath *from* u *to* w *if* P *is the disjoint union of a path from* u *to* w *and some number (perhaps 0) of cycles. In other words, if* v *is a vertex of* P*, then*

$$d_P^+(v) = \begin{cases} 0 & \text{if } v = w; \\ 1 & \text{otherwise}; \end{cases} \qquad \text{and} \qquad d_P^-(v) = \begin{cases} 0 & \text{if } v = u; \\ 1 & \text{otherwise}. \end{cases}$$

Definition 3.6 *Let* u, w *be integers representing vertices of* $\mathrm{Circ}(6m; 2, 3, c)$*. If* $u + c' + 2 \leq w \leq u + 2c'$*, let* $P(u, w)$ *be the pseudopath from* $u+1$ *to* $w-1$ *whose vertex set is* $I(u, w)$*, such that* v *travels by*

$$\begin{cases} 2, & \text{if } v \in I(u, w - c' - 3) \cup I(u + c' + 1, w - 2), \\ 3, & \text{if } v \in I(w - c' - 2, u + c' - 1), \\ -c', & \text{if } v \in \{u + c', w\}. \end{cases}$$

Notice that the range of values for w *because* $c' > 3$.

Lemma 3.7 $P(u, w)$ *is a path if any of the following hold:*

- $w - u \equiv 2c' \pmod 3$*; or*
- $w - u \equiv 2c' + 1 \pmod 3$ *and* $w - u \equiv c' \pmod 2$*; or*
- $w - u \equiv 2c' + 2 \pmod 3$ *and* $w - u \not\equiv c' \pmod 2$.

PROOF. We may assume that $u = 0$. Let $\varepsilon \in \{1, 2\}$ be such that $w - c' - \varepsilon - 1$ is even.

When $w \equiv 2c' \pmod 3$, the path in $P(0,w)$ is

$$\begin{aligned} &1 \overset{2}{\dashrightarrow} w-c'-\varepsilon \overset{3}{\dashrightarrow} c'-\varepsilon+3 \overset{2}{\dashrightarrow} w \overset{-c'}{\longrightarrow} w-c' \\ &\overset{3}{\dashrightarrow} c' \overset{-c'}{\longrightarrow} 0 \overset{2}{\dashrightarrow} w-c'+\varepsilon-3 \overset{3}{\dashrightarrow} c'+\varepsilon \overset{2}{\dashrightarrow} w-1. \end{aligned}$$

When $w \not\equiv 2c' \pmod 3$, the hypothesis of the Lemma implies that $w \equiv 2c'+\varepsilon \pmod 3$. In this case, the path in $P(0,w)$ is

$$\begin{aligned} &1 \overset{2}{\dashrightarrow} w-c'-\varepsilon \overset{3}{\dashrightarrow} c' \overset{-c'}{\longrightarrow} 0 \overset{2}{\dashrightarrow} w-c'+\varepsilon-3 \\ &\overset{3}{\dashrightarrow} c'-\varepsilon+3 \overset{2}{\dashrightarrow} w \overset{-c'}{\longrightarrow} w-c' \overset{3}{\dashrightarrow} c'+\varepsilon \overset{2}{\dashrightarrow} w-1. \end{aligned}$$

In both cases, it can be verified that the path from 1 to $w-1$ contains all vertices in $I(0,w)$, and thus $P(0,w)$ is a path. (Note that it suffices to check that the path contains both c-arcs, for then $P(0,w)$ cannot contain any cycles.) □

Lemma 3.8 *Let $k \in \mathbb{Z}$ be such that*

- $k \le 6m$,
- $c'+3 \le k \le 2c'+2$, *and*
- $k+c' \not\equiv 3 \pmod 6$.

Let u,w be integers representing vertices of $\mathrm{Circ}(6m;2,3,c)$. *If $u \le w$ and $w-u+1=k$, then the subgraph induced by $I(u,w)$ has a hamiltonian path that starts at $u+1$ and ends in $\{w-1,w\}$.*

PROOF. We consider three cases.

Case 1 *Assume $k \equiv 2c'+1 \pmod 3$.*

We have $w-u=k-1 \equiv 2c' \pmod 3$. Since $k \le 2c'+2$, this implies $w-u \le 2c'$. By Lemma 3.7, $P(u,w)$ is a hamiltonian path from $u+1$ to $w-1$.

Case 2 *Assume $k \equiv 2c'+2 \pmod 3$.*

Suppose, first, that $k \ne c'+3$ (so $k \ge c'+4$). Letting $w'=w-1$, then

$$w'-u = w-u-1 = k-2 \ge (c'+4)-2 = c'+2$$

and $w'-u=(w-1)-u=k-2 \equiv 2c' \pmod 3$. By Lemma 3.7, $P(u,w')$ is a hamiltonian path in $I(u,w')$ from $u+1$ to $w'-1$. Adding the 2-arc from $w'-1$ to $w'+1=w$ yields a hamiltonian path in $I(u,w)$ from $u+1$ to w.

Suppose instead that $k = c' + 3$. Then $w - u = k - 1 \equiv 2c' + 1 \pmod 3$ and $w - u = k - 1 = (c' + 3) - 1 \equiv c' \pmod 2$, so by Lemma 3.7, $P(u, w)$ is a hamiltonian path from $u + 1$ to $w - 1$.

Case 3 *Assume $k \equiv 2c' \pmod 3$.*

By assumption, we have $k + c' \equiv 2c' + c' \equiv 0 \pmod 3$. Since $k + c' \not\equiv 3 \pmod 6$, we must have $k + c' \equiv 0 \pmod 6$, so $k \equiv c' \pmod 2$. Then $w - u = k - 1 \equiv 2c' + 2 \pmod 3$ and $w - u \equiv k - 1 \not\equiv k \equiv c' \pmod 2$, so by Lemma 3.7, $P(u, w)$ is a hamiltonian path from $u + 1$ to $w - 1$. $\square$

It is now easy to prove Propositions 3.1 and 3.2.

PROOF OF PROPOSITION 3.1. As previously mentioned, we may assume $c > 3$. Since $3 < c \leq 3m$, we have $3m \leq c' < 6m - 3$, so

$$c' + 3 < 6m < 2c' + 2.$$

Furthermore, since $c \not\equiv 3 \pmod 6$, we have

$$6m + c' \equiv c' \not\equiv 3 \pmod 6.$$

Hence, Lemma 3.8 implies that the interval $I(0, 6m - 1)$ has a hamiltonian path from 1 to $6m - 2$ or to $6m - 1$. Inserting the 3-arc from $6m - 2$ to 1 or the 2-arc from $6m - 1$ to 1, yields a hamiltonian cycle. Since $I(0, 6m - 1)$ is the entire digraph, this completes the proof. $\square$

Lemma 3.9 *Let $\mathcal{K}$ be the set of integers k that satisfy the conditions of Lemma 3.8. If $4 \leq c' < 3m$, and $n_0 \geq 2(c' + 4)$, then either*

(1) *n_0 can be written as a sum $n_0 = k_1 + k_2 + \cdots + k_s$, with each $k_i \in \mathcal{K}$, or*
(2) *$c' = 6$ and $n_0 = 29$.*

PROOF. Note that, since $c' < 3m$, we have $2c' + 2 \leq 6m$, so the first inequality in the definition of $\mathcal{K}$ is redundant — it can be ignored.

Let us treat some small cases individually:

- If $c' = 4$, then $\mathcal{K} = \{7, 8, 9, 10\}$. It is easy to see that every integer ≥ 14 is a sum of elements of $\mathcal{K}$.
- If $c' = 5$, then $\mathcal{K} = \{8, 9, 11, 12\}$. It is easy to see that every integer ≥ 16 is a sum of elements of $\mathcal{K}$.
- If $c' = 6$, then $\mathcal{K} = \{10, 11, 12, 13, 14\}$. It is easy to see that every integer ≥ 20 is a sum of elements of $\mathcal{K}$, except that 29 is not such a sum.

Henceforth, we assume $c' \geq 7$, so $4c' + 5 \geq 3(c' + 4)$. Then, since $c' + 4 \in \mathcal{K}$, we may assume, by subtracting some multiple of $c' + 4$, that

$$n_0 \leq 3(c' + 4) - 1 \leq 4c' + 4.$$

Under this assumption, we prove the more precise statement that

$$n_0 = k_1 + k_2, \text{ with } k_1, k_2 \in \mathcal{K}, \text{ and } k_1 \leq k_2 \leq k_1 + 3.$$

Assume that n_0 cannot be written as such a sum. (This will lead to a contradiction.) Because $2(c'+4) = (c'+4)+(c'+4)$, we must have $n_0 > 2(c'+4)$. Then, by induction, we may assume there exist $k_1, k_2 \in \mathcal{K}$, such that $k_1 \leq k_2 \leq k_1+3$ and

$$k_1 + k_2 = n_0 - 1.$$

Since $2c' + 1, 2c' + 2 \in \mathcal{K}$, we must have

$$n_0 \leq 4c' + 1,$$

so

$$k_1 \leq 4c'/2 = 2c'.$$

Note that

$$\begin{aligned} n_0 &= (k_1 + 1) + k_2, \\ n_0 &= (k_1 + 2) + (k_2 - 1), \\ n_0 &= k_1 + (k_2 + 1). \end{aligned}$$

From the first equation (and the fact that $k_2 \in \mathcal{K}$, we see that $k_1 + 1 \notin \mathcal{K}$. Because $\mathcal{K}$ contains 5 of any 6 consecutive integers between $c' + 3$ and $2c' + 2$, this implies that $k_1 + 2 \in \mathcal{K}$. Hence, the second equation implies $k_2 - 1 \notin \mathcal{K}$. Therefore $k_2 - 1 = k_1 + 1$, so

$$k_2 + 1 = k_1 + 3.$$

Hence, the third equation implies $k_2+1 \notin \mathcal{K}$, so we must have $k_2+1 > 2c'+2$; therefore $k_2 = 2c' + 2$, which implies $k_1 = 2c'$. Hence

$$4c' + 2 = k_1 + k_2 = n_0 - 1 \leq 4c'.$$

This is a contradiction. □

PROOF OF PROPOSITION 3.2. Note that $c' < 3m$. We dealt with the cases $c = 6m - 2$ and $c = 6m - 3$ in Lemma 2.5, and the cases $6m \in \{2c' + 4, 2c' + 6\}$ are dealt with by Theorem 1.2. Furthermore, we noted in Assumption 2.2 that the case $c = 6m - 1$ is clearly hamiltonian. Therefore, we may assume in what follows that $c' \geq 4$ and $6m \notin \{2c' + 4, 2c' + 6\}$.

Let $\mathcal{K}$ be the set of integers k that satisfy the conditions of Lemma 3.8. We claim that $6m$ can be written as a sum $6m = k_1 + k_2 + \cdots + k_s$, with each $k_i \in \mathcal{K}$. If $6m \geq 2(c' + 4)$, then this is immediate from Lemma 3.9 (and the fact that $6m \neq 29$). On the other hand, if $6m < 2(c' + 4)$, then, since $6m$ is even, and $6m \notin \{2c' + 4, 2c' + 6\}$, we see that $6m = 2c' + 2 \in \mathcal{K}$, so $6m$ is obviously a sum of elements of $\mathcal{K}$. This completes the proof of the claim.

The preceding paragraph implies that we may cover the vertices of $\mathrm{Circ}(6m; 2, 3, c)$ by a disjoint collection of intervals $I(u_i, w_i)$, such that the number of vertices in $I(u_i, w_i)$ is k_i. By listing the intervals in their natural order, we may assume $u_{i+1} = w_i + 1$. By Proposition 3.8, the vertices of $I(u_i, w_i)$ can be covered by a path P_i that starts at $u_i + 1$ and ends in $\{w_i - 1, w_i\}$. Since

$$(u_{i+1} + 1) - w_i = (w_i + 2) - w_i = 2$$

and

$$(u_{i+1} + 1) - (w_i - 1) = (w_i + 2) - (w_i - 1) = 3,$$

there is an arc from the terminal vertex of P_i to the initial vertex of P_{i+1}. Thus, by adding a number of 2-arcs and/or 3-arcs, we may join all of the paths $P_1, P_2, \ldots, P_s$ into a single cycle that covers all of the vertices of $\mathrm{Circ}(6m; 2, 3, c)$. Thus, we have constructed a hamiltonian cycle. $\square$

4 The remaining cases

In this section, we prove the following result.

Proposition 4.1 *If $3 < c \leq 3m$ and $c \equiv 3 \pmod 6$, then $\mathrm{Circ}(6m; 2, 3, c)$ has a hamiltonian cycle.*

Proposition 4.1 together with Propositions 3.2 and 3.1 (and Theorem 1.2) completes the proof of Theorem 1.3.

Definition 4.2 *Let t be any natural number, such that $0 \leq 6t \leq c - 9$.*

(1) *Let*

$$\begin{aligned} \ell_1 &= c - 5, \\ \ell_2 = \ell_2(t) &= c - 1 + 6t, \\ \ell_3 &= c - 2, \\ \ell_4 &= c + 3. \end{aligned}$$

(2) *Define subdigraphs Q_1, Q_2, Q_3 and Q_4 of $\mathrm{Circ}(6m; 2, 3, c)$ as follows:*
- *The vertex set of Q_i is $I(0, \ell_i + 2) \cup \{\ell_i + 5\}$.*

- *In Q_1, vertex v travels by* $\begin{cases} c, & \text{if } v = 0; \\ 2, & \text{if } v = 1 \text{ or } 2; \\ 3, & \text{if } v = 3, 4, \ldots, c-6. \end{cases}$
- *In Q_2, vertex v travels by* $\begin{cases} c, & \text{if } v = 1 \text{ or } 6t+4; \\ 2, & \text{if } v = 2 \text{ or } 6t+5 \leq v \leq c-2; \\ 3, & \text{if } v = 0 \text{ or } 3 \leq v \leq 6t+3 \text{ or } c-1 \leq v \leq c-2+6t. \end{cases}$
- *In Q_3, vertex v travels by* $\begin{cases} c, & \text{if } v = 1 \text{ or } 3; \\ 2, & \text{if } v = 1, 2 \text{ or } 4 \leq v \leq c-3. \end{cases}$
- *In Q_4, vertex v travels by* $\begin{cases} c, & \text{if } v = 2 \text{ or } 8; \\ 2, & \text{if } 9 \leq v \leq c-1; \\ 3, & \text{if } v = 0, 1 \text{ or } 3 \leq v \leq 7 \text{ or } c \leq v \leq c+2. \end{cases}$

Notation 4.3 *For ease of later referral, we also let $\ell_i(t)$ denote ℓ_i for $i \in \{1, 3, 4\}$.*

Lemma 4.4

(1) *Each Q_i is the union of four disjoint paths from $\{0, 1, 2, 5\}$ to $\{\ell_i, \ell_i + 1, \ell_i + 2, \ell_i + 5\}$.*
(2) *More precisely, let $u_1 = 0$, $u_2 = 1$, $u_3 = 2$, and $u_4 = 5$, and define permutations*

$$\sigma_1 = (1423),\ \sigma_2 = (234),\ \sigma_3 = (1324),\ \textit{and } \sigma_4 = \text{identity}.$$

Then Q_i contains a path from u_k to $\ell_i + u_{\sigma_i(k)}$, for $k = 1, 2, 3, 4$.

PROOF. The paths in Q_1 are:

$$\begin{aligned}
&0 \xrightarrow{c} c \quad (= \ell_1 + 5), \\
&1 \xrightarrow{2} 3 \overset{3}{\dashrightarrow} c-3 \quad (= \ell_1 + 2), \\
&2 \xrightarrow{2} 4 \overset{3}{\dashrightarrow} c-5 \quad (= \ell_1), \\
&5 \overset{3}{\dashrightarrow} c-4 \quad (= \ell_1 + 1).
\end{aligned}$$

The paths in Q_2 are:

$$\begin{aligned}
&0 \overset{3}{\dashrightarrow} 6t+6 \overset{2}{\dashrightarrow} c-1 \overset{3}{\dashrightarrow} c-1+6t \quad (= \ell_2), \\
&1 \xrightarrow{c} c+1 \overset{3}{\dashrightarrow} c+1+6t \quad (= \ell_2 + 2), \\
&2 \xrightarrow{2} 4 \overset{3}{\dashrightarrow} 6t+4 \xrightarrow{c} c+4+6t \quad (= \ell_2 + 5), \\
&5 \overset{3}{\dashrightarrow} 6t+5 \overset{2}{\dashrightarrow} c \overset{3}{\dashrightarrow} c+6t \quad (= \ell_2 + 1).
\end{aligned}$$

The paths in Q_3 are:

$$
\begin{aligned}
&0 \xrightarrow{c} c \quad (= \ell_3 + 2),\\
&1 \xrightarrow{2} 3 \xrightarrow{c} c+3 \quad (= \ell_3 + 5),\\
&2 \overset{2}{\dashrightarrow} c-1 \quad (= \ell_3 + 1),\\
&5 \overset{2}{\dashrightarrow} c-2 \quad (= \ell_3).
\end{aligned}
$$

The paths in Q_4 are:

$$
\begin{aligned}
&0 \overset{3}{\dashrightarrow} 9 \overset{2}{\dashrightarrow} c \xrightarrow{3} c+3 \quad (= \ell_4),\\
&1 \overset{3}{\dashrightarrow} 10 \overset{2}{\dashrightarrow} c+1 \xrightarrow{3} c+4 \quad (= \ell_4 + 1),\\
&2 \xrightarrow{c} c+2 \xrightarrow{3} c+5 \quad (= \ell_4 + 2),\\
&5 \xrightarrow{3} 8 \xrightarrow{c} c+8 \quad (= \ell_4 + 5). \qquad \square
\end{aligned}
$$

From the above lemma, we see that translating Q_j by ℓ_i yields a disjoint union of 4 paths whose initial vertices are precisely the terminal vertices of the paths in Q_i. Hence, composing Q_i with this translate of Q_j results in a disjoint union of 4 paths: namely, a path from u_k to $\ell_i + \ell_j + u_{\sigma_j \sigma_i(k)}$, for $k = 1, 2, 3, 4$. Continuing this reasoning leads to the following conclusion:

Lemma 4.5 *If, for some natural number s, there exist sequences*

- $I = (i_1, i_2, \ldots, i_s)$ *with each* $i_j \in \{1, 2, 3, 4\}$, *and*
- $T = (t_1, t_2, \ldots, t_s)$ *with* $0 \le 6t_j \le c - 9$, *for each* j,

such that

(i) $\sum_{j=1}^{s} \ell_{i_j}(t_j) = 6m$, *and*
(ii) *the permutation product* $\sigma_{i_s}\sigma_{i_{s-1}} \cdots \sigma_{i_1}$ *is a cycle of length* 4,

then $\mathrm{Circ}(6m; 2, 3, c)$ *has a hamiltonian cycle constructed by concatenating s appropriate translates of* Q_1, Q_2, Q_3, *and/or* Q_4.

PROOF OF PROPOSITION 4.1. Since σ_4 is the identity and $\ell_4 = c+3$, we see that if $\mathrm{Circ}(6m; 2, 3, c)$ has a hamiltonian cycle constructed by concatenating translates of Q_1, Q_2, Q_3, and Q_4, then $\mathrm{Circ}(6m + c + 3; 2, 3, c)$ also has such a hamiltonian cycle. Thus, by subtracting some multiple of $c+3$ from $6m$, we may assume

$$2c - 6 \le 6m \le 3c - 9.$$

(For this modified c, it is possible that $c > 3m$.)

Recall that $0 \leq 6t \leq c - 9$, so $2c - 6 + 6t$ can be any multiple of 6 between $2c - 6$ and $3c - 15$. Since $\sigma_2\sigma_1 = (1243)$ and $\ell_1 + \ell_2(t) = 2c - 6 + 6t$, it follows that $\mathrm{Circ}(6m; 2, 3, c)$ has a hamiltonian cycle constructed by concatenating Q_1 with a translate of Q_2 whenever $2c - 6 \leq 6m \leq 3c - 15$.

The only case that remains is when $6m = 3c - 9$. Now $\sigma_3^2\sigma_1 = (1324)$ and $\ell_1 + 2\ell_3 = 3c - 9$, so $\mathrm{Circ}(3c - 9; 2, 3, c)$ has a hamiltonian cycle constructed by concatenating Q_1 with two translates of Q_3. $\square$

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